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# THE FAILURE OF A BOUNDARY LAYER MODEL TO DESCRIBE CERTAIN CASES OF CELLULAR CONVECTION

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**Abstract**—A model has been developed to describe the two-dimensional cellular motion of a viscous fluid with Prandtl number of order unity heated from below at high Rayleigh number. This model is easily adapted to describe such flow for infinite Prandtl number provided all boundaries are free. It has been suggested that a good approximation to the physical problem of flow between rigid horizontal boundaries will be obtained by setting the Prandtl number equal to infinity, thus ignoring the momentum convection. Similar cellular motion occurs in flow in a porous medium at high Rayleigh numbers and a similar model has been proposed to describe this motion. The basic features of the model are outlined and it is shown that the model fails to describe either the motion of an infinite Prandtl number fluid between parallel rigid boundaries or fluid in a porous medium. An analysis of the model for large finite Prandtl number and large Rayleigh number shows that as the Prandtl number is increased the velocity boundary layers on the rigid horizontal boundaries thicken, and eventually fill the cell, thus losing their boundary layer identity and causing the breakdown of the model.

## NOMENCLATURE

$A$, Rayleigh number for flow in a porous medium;
$BL$, subscript denoting boundary layer function;
$d$, vertical separation of the plates;
$int$, subscript denoting interior function;
$K$, permeability;
$L$, cell width;
$Nu$, Nusselt number;
$Pr$, Prandtl number;
$Ra$, Rayleigh number;
$T_0$, average temperature;
$\Delta T$, temperature difference between the plates;
$\mathbf{u}$, velocity vector;
$x$, horizontal coordinate measured from one cell boundary;
$z$, vertical coordinate measured from the lower plate.

Greek symbols
$\alpha$, coefficient of thermal expansion;
$\gamma$, coefficient of volumetric expansion;
$\delta, \delta_H$, order of magnitude width of horizontal velocity boundary layers;
$\delta_T$, order of magnitude width of horizontal thermal boundary layers, introduced only when $\delta_H \neq \delta_T$;
$\delta_v$, order of magnitude width of vertical boundary layers;
$\theta$, scaled temperature difference;
$\eta$, vorticity;
$\kappa$, thermometric conductivity;
$\kappa_m$, diffusivity of a fluid in a porous medium;
$\nu$, kinematic viscosity;
$\psi$, stream function;
[ ], square brackets denote order of magnitude of a quantity.

## INTRODUCTION

WHEN a fluid is heated from below it becomes unstable at a critical Rayleigh number and when the Rayleigh number is just above this critical value the flow takes the form of steady convection cells. An asymptotic expansion has been developed to describe this flow for values of the

Rayleigh number just above critical (for a review of this work see Segel [1]). When the Rayleigh number is increased further the flow consists of two dimensional laminar convection cells, as has been shown by the careful experiments of Koshmeider [2]. For very large Rayleigh numbers the experiments of Rossby [3] show that the flow becomes time-dependent and eventually turbulent. It seems reasonable therefore to study a model of two-dimensional laminar convection cells for high Rayleigh numbers both to understand this flow and hopefully to explain the onset of time-dependence and turbulence.

The experiments of Elder [4] show that the flow in a porous medium at high Rayleigh number also takes the form of laminar convection cells.

A boundary layer model has been developed by Robinson [5] to describe the motion of a fluid with Prandtl number of order unity heated from below at high Rayleigh number. Rectangular cells with both rigid and free boundary conditions were considered. This model may be easily modified to describe such motion for a fluid with an infinite Prandtl number provided that all boundaries are free (see Robinson [6, 7], Turcotte and Oxburg [8]) and this extension of the model and the results of a simple computation are outlined in an appendix to this paper.

The model has been used by Weinbaum [9] to describe motion in a rigid horizontal cylinder for Prandtl number of order unity and by Menold and Ostrach [10] to describe motion in a rigid horizontal cylinder for infinite Prandtl number. The latter application of the model is questionable as no velocity boundary layer exists near the top and bottom of the cylinder and there is insufficient velocity to convect the heat along the boundary layer at those places. It is suggested that a two-cell model may be appropriate for this problem.

It has been suggested by several authors that a good approximation to the physical problem of flow between rigid horizontal boundaries is obtained by setting the Prandtl number equal to infinity, thus ignoring the momentum convection. It is therefore of interest to determine whether the above model may be used to describe either the motion of an infinite Prandtl number fluid between parallel horizontal rigid boundaries as has been suggested by Turcotte [11] or the flow in a porous medium as has been suggested by Elder [4]. We discuss here the essential features of this model and show that it fails to describe the flow in either case.

We refer to "infinite Prandtl number" when $1 \ll R \ll Rr^{\frac{3}{2}}$ (see Robinson [5], p. 598). If the $\frac{3}{2}$ power of the Prandtl number is small compared with the Rayleigh number (even if it is large compared to unity) then the model developed by Robinson [5] is appropriate and the variation of the heat flux with change in Prandtl number predicted by the theory agrees well with the available experimental results (*loc. cit.*, Fig. 8).

The available numerical results on two-dimensional convection cells at high Rayleigh number [12–14] all consider Prandtl numbers of order unity.

## OUTLINE OF THE MODEL AND APPLICATION TO THE MOTION IN AN INFINITE PRANDTL NUMBER FLUID BETWEEN RIGID HORIZONTAL BOUNDARIES

The Boussinesq approximation is used in writing the equations for conservation of mass, momentum (and eventually vorticity) and thermal energy. The density and the coefficients of viscosity and thermal diffusivity are assumed to be constant except for the density in the body force term in the momentum and vorticity equations.

With the non-dimensionalization $\mathbf{r} = d\mathbf{r}'$, $t = (d^2/\kappa)\, t'$, $T = T_0 + \Delta T\theta$ where $d$ is the separation of the horizontal plates, $\kappa$ is the thermometric conductivity, $\Delta T$ is the temperature difference between the plates and $T_0$ is the mean temperature, the equations of motion in two dimensions are

$$(\mathbf{v}\,.\,\nabla)\,\theta = -\frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial z} = \nabla^2\theta \qquad (1)$$

$$Pr^{-1}(\mathbf{v}\,.\,\nabla)\,\eta = \nabla^2\eta - Ra\frac{\partial\theta}{\partial x}. \qquad (2)$$

$\psi$ is the stream function of flow [$u_x = -(\partial\psi/\partial z)$, $u_z = (\partial\psi/\partial x)$; the continuity equation $(\partial u_x/\partial x) + (\partial u_z/\partial z) = 0$ is automatically satisfied]; $\eta$ is the vorticity $= (\partial u_x/\partial z) - (\partial u_z/\partial x) = -\nabla^2\psi$; $Ra$ is the Rayleigh number $Ra = (g\alpha\Delta T d^3/\kappa\nu)$; $Pr = \nu/\kappa$ is the Prandtl number of the fluid.

For infinite Prandtl number the vorticity equation (2) becomes

$$\nabla^2\eta = Ra\frac{\partial\theta}{\partial x}. \qquad (3)$$

We will discuss here rectangular convection cells with dimensionless width $L$. The fluid is confined between rigid horizontal boundaries and is heated from below. The temperature boundary conditions are $\theta = \mp\frac{1}{2}$ on $z = {}^{1}_{0}$. The vertical boundaries are periodic—we consider one of an infinite set of cells—and the temperature boundary conditions on the vertical boundaries are $\partial\theta/\partial x = 0$ (on $x = 0, L$). These are also the boundary conditions appropriate for insulated vertical boundaries.

The velocity component perpendicular to the boundaries is zero, and thus we set $\psi = 0$ on all boundaries ($z = 0$, 1 and $x = 0$, $L$). The horizontal boundaries are rigid and the velocity parallel to these boundaries vanishes, i.e., $u_x = -(\partial\psi/\partial z) = 0$ on $z = 0$, 1. Since the vertical boundaries between the cells are periodic there must be zero stress along these boundaries, i.e. $(\partial u_z/\partial x) = -(\partial^2\psi/\partial x^2) = 0$ on $x = 0$, $L$. The stream function is constant along the boundaries giving $(\partial^2\psi/\partial z^2) = 0$ on $x = 0$, $L$ and this boundary condition can be written $-\nabla^2\psi = \eta = 0$.

The temperature and the stream function are written as an interior function plus boundary layer functions

$$\psi = \psi_{int} + \sum \psi_{BL}$$
$$\theta = \theta_{int} + \sum \theta_{BL}.$$

All boundary layer functions depend on the boundary layer coordinate and tend to zero in the interior of the cell. It is the sums of these interior and boundary layer functions which must satisfy the boundary conditions.

It is assumed that the streamlines for the inner flow are closed. Since the interior velocity is always found to be greater than unity by an order of magnitude the heat conduction is unimportant in the interior and the temperature is convected unchanged around the (closed) streamlines. Since the problem is asymmetrical this constant temperature is positive near the lower boundary and negative near the upper boundary. The interior temperature must therefore be zero: $\theta_{int} = 0$. The equation for the interior stream function is then

$$\nabla^4\psi_{int} = 0. \qquad (4)$$

The temperature difference, $\theta$, is assumed to be non-zero in boundary layers along each wall, and all heat transport takes place in those boundary layers.

The following are the balances that must be satisfied in the boundary layers:

(1) The flow is driven by the buoyancy forces acting in the vertical boundary layers. The vorticity creation and conduction terms in the vorticity equation (3) must be of equal order of magnitude in those boundary layers:

$$\nabla^2\eta_{BL} = \frac{\partial^2\eta_{BL}}{\partial x^2} \sim [\eta_{BL}]\,\delta_v^{-2}$$

and

$$Ra\frac{\partial\theta}{\partial x} \sim Ra\,\delta_v^{-1}.$$

This balance requires that

$$Ra\,\delta_v \sim [\eta_{BL}] \qquad (5)$$

Square brackets denote the order of magnitude of a quantity, $\delta_v$ is the width of the vertical boundary layers and $\delta_H$ is the width of the horizontal boundary layers. The subscript $BL$ denotes a boundary layer function and the subscript *int* an interior function.

(2) The heat is conducted through the horizontal boundaries and convected away in the horizontal boundary layers. The velocity in these regions is the interior velocity; that is we take the interior velocity to satisfy both boundary conditions on the horizontal boundaries. The velocity parallel to the boundaries is thus

$$u_{\parallel} \approx \frac{\partial u_{int}}{\partial z} z \sim [u_{int}]\,\delta_H.$$

The heat conduction is $u_{\parallel}\delta_H \sim [u_{int}]\,\delta_H^2$. The heat convection through the boundaries is $\int_0^L (\partial\theta/\partial z)\,dx \sim \delta_H^{-1}$. The two terms may balance provided

$$[u_{int}] = \delta_H^3 \tag{6}$$

This is equivalent to the requirement that the heat convection and conduction terms should be of equal order of magnitude in the horizontal boundary layers.

(3) The heat which is conducted into the cell through the horizontal boundaries is convected along the vertical boundary layers

$$\delta_H^{-1} = [u_{int}]\,\delta_v. \tag{7}$$

(4) The boundary layer function in the vertical boundary layers are chosen such that the necessary boundary conditions may be satisfied. Thus $\eta_{BL} + \eta_{int} = 0$ on $x = 0, L$. Since the interior functions vary over distances of order unity,

$$[u_{int}] = [\eta_{int}] = [\eta_{BL}]. \tag{8}$$

The vorticity equation in those regions may be integrated to give $\eta_{BL} = Ra\int\theta dx$ where $\eta_{BL}$ is the value of the boundary layer vorticity on the boundary and the temperature integration is taken across the boundary layer. The boundary condition on the interior stream function is then $\nabla^2\psi_{int} = Ra\int\theta dx$ on that boundary.

Equations (5)–(8) give us four relations from which we may determine the four unknowns. They are

$$[u_{int}] = Ra^{\frac{3}{5}}$$

$$\delta_v = Ra^{-\frac{2}{5}}$$

$$\delta_H = Ra^{-\frac{1}{5}}$$

$$[\eta_{BL}] = Ra^{\frac{3}{5}},$$

and the Nusselt number of the flow has magnitude $[Nu] = Ra^{\frac{1}{5}}$.

The model is self-consistent provided $(\mathbf{v}\nabla) \gg \nabla^2\theta$ in the vertical boundary layers. Otherwise the heat will diffuse faster than it is transported and the boundary layer will lose its identity, i.e. the temperature equation in those regions would be $(\partial^2\theta/\partial x^2) = 0$, which does not have a boundary layer type solution (with $\theta \neq 0$, $(\partial\theta/\partial x) = 0$ and $\theta \to 0$ as $x \to \infty$).

In the model above we find $(\mathbf{v}\,.\,\nabla)\,\theta \sim [u_{int}] = Ra^{\frac{3}{5}}$ and $\nabla^2\theta \sim \delta_v^{-2} = Ra^{\frac{4}{5}}$. This necessary condition that the model be self-consistent is not satisfied and the model cannot describe a possible motion of the system.

The model may be altered by postulating velocity boundary layers near the vertical boundaries (rather than the vorticity boundary layers demanded by the boundary conditions), or velocity boundary layers near the horizontal boundaries, or both, but in each case an inconsistency arises.

If the Prandtl number is of order unity the vorticity convection may not be neglected. The vorticity equation is then equation (2) above and the required balances are satisfied by $[u_{int}] = Ra^{\frac{2}{3}}$, $\delta_v = \delta_H = Ra^{\frac{1}{3}}$, $[Nu] = Ra^{\frac{1}{3}}$ with constant interior vorticity and velocity boundary layers along the horizontal boundaries in which there is a balance between the momentum convection and the conduction terms. It is these boundary layers which allow the velocity to be of sufficient order of magnitude to convect the necessary heat away from the horizontal boundaries; when the convection term is neglected as it is in the infinite Prandtl number approximation no such velocity layers may exist. If, however, the boundaries are all free the internal velocity may have a non-zero component parallel to these boundaries and the heat transfer may be achieved without the aid of velocity boundary layers. When the boundary is a rigid horizontal cylinder there are regions at the top and bottom where the boundaries are horizontal and there is again insufficient thermal convection in the boundary layer. The model, which has been applied by Menold and Ostrach [10] in an attempt to describe this flow, again fails. It is suggested that a two-cell model may be appropriate in this case.

Turcotte [9] has suggested a similar model to describe this flow. His analysis suggests $[u_{int}] = R^{\frac{3}{4}}$ and $\delta_v = \delta_H = R^{\frac{1}{4}}$. Using these values, the heat conducted through the horizontal boundaries is of order $\delta_H^{-1} = R^{\frac{1}{4}}$, and the heat convected along the vertical boundary layers is of order $[u_{int}]\,\delta_v = R^{\frac{1}{2}}$. Thus the necessary balance between the heat conducted through the horizontal boundaries and the heat convected along the vertical boundary layers is not satisfied in that model.

## APPLICATION TO FLOW IN A POROUS MEDIUM

The experiments of Elder [4] show that a similar cellular motion may be expected for flow in a porous medium; it is therefore of interest to determine whether this model will describe that flow at large Rayleigh number.

The vorticity equation for flow in a porous medium has been shown by Wooding [15] to be

$$\eta = A\,.\,\theta_x \tag{9}$$

where $A = gK\gamma\Delta Td/\kappa_m\nu$ is the Rayleigh number of this problem. $K$ is the permeability, $\kappa_m$ the diffusivity and $\gamma$ the volumetric coefficient of expansion. The normal velocity vanishes on all boundaries so we set $\psi = 0$ on $x = 0, L$ and $z = 0, 1$. The temperature equation and boundary conditions are unchanged.

We again assume that the interior streamlines are closed and find that the interior functions satisfy the equations

$$\theta_{int} = 0 \qquad \eta_{int} = 0.$$

The necessary balances are similar to those given above for an infinite Prandtl number viscous fluid. They are:

(1) The flow is driven by the buoyancy forces in the vertical boundary layers. The vorticity in those boundary layers must balance the temperature gradient

$$[\eta_{BL}] = \delta_v^{-1}. \tag{10}$$

(2) The heat is conducted through the horizontal boundaries and convected away in the horizontal boundary layers.

$$[u_{int}]\,\delta_H = \delta_H^{-1}. \tag{11}$$

(3) The heat which is conducted into the horizontal boundary layers is convected along the vertical boundary layers. The fourth condition below implies that the vertical boundary layer velocity will be greater in order of magnitude than the interior velocity.

$$[u_{BL}]\,\delta_v = \delta_H^{-1}. \tag{12}$$

(4) The boundary layer function in the vertical boundary layers are chosen such that the necessary boundary conditions may be satisfied. Thus $\psi_{int} + \psi_{BL} = 0$ on $x = 0, L$ and

$$[\psi_{BL}] = [u_{BL}]\,\delta_v = [\eta_{BL}]\,\delta_v^2 = [u_{int}]. \tag{13}$$

If the vorticity boundary layer equation is integrated across the boundary layer, the boundary condition on the interior stream function is found to be $\psi_{int} = -\psi_{BL} = -A\int\theta dx$.

Equations (11–13) may be solved to give $\delta_H = 1$ which is contrary to the boundary layer assumption of the model. The model cannot therefore describe this flow.

Elder [4] presented experimental and computed results for flow in a porous medium which suggests that for large Rayleigh number the Nusselt number is proportional to the Rayleigh number. He gives a short analysis of a horizontal boundary layer which suggests $\delta_H = A^{-1}$; $[\partial\psi_{BL}/\partial x] = A$. However in this model

$$[\partial^2\psi_{BL}/\partial z^2] = [\psi_{BL}]\,\delta_H^{-2} = A^3$$

in this region, and the vorticity equation is $\partial^2\psi/\partial z^2 = 0$, which does not have a boundary layer type solution. This short analysis cannot therefore form the basis of a complete model of the flow.

### APPLICATION TO MOTION OF A VISCOUS FLUID WITH LARGE RAYLEIGH NUMBER

It is instructive to examine the variation of the boundary layer thickness in a large Prandtl number fluid in order to see how the breakdown of the model occurs as the Prandtl number increases to infinity. For finite Prandtl number the vorticity equation is equation (2) above. The derivation of the interior solutions is as outlined by Robinson [7]; the interior equations are $\nabla^2 \psi_{int} = \omega_0$, $\theta_{int} = 0$.

The velocity and temperature boundary layers on the horizontal boundaries will now have different thicknesses, $\delta_H$, $\delta_T$ with $\delta_H \gg \delta_T$. The balances of the model are as follows:

(1) The flow is driven by the buoyancy forces in the vertical boundary layers. For large Prandtl number there is a balance between vorticity creation and conduction in these regions

$$[\eta_{BL}]\,\delta_v^{-2} = Ra\,\delta_v^{-1}. \tag{14}$$

(2) The heat is conducted through the horizontal boundary layers. Since these layers are narrower than the velocity boundary layers, the appropriate velocity parallel to the boundaries will be of order of magnitude $[u_{int}]\,(\delta_T/\delta_H)$. The heat flux balance is thus

$$[u_{int}]\,(\delta_T/\delta_H)\,\delta_T = \delta_T^{-1}. \tag{15}$$

(3) In the velocity boundary layers along the rigid horizontal boundaries there will be a balance between the momentum conduction and convection terms

$$(1/Pr)\,[u_{int}] = \delta_H^{-2}. \tag{16}$$

(4) All the heat flux is convected along the vertical boundary layers.

$$[u_{int}]\,\delta_v = \delta_T^{-1}. \tag{17}$$

(5) The boundary layer vorticity in the vertical boundary layers is chosen to satisfy the boundary condition $\eta_{int} + \eta_{BL} = 0$.

$$[\eta_{BL}] = [u_{int}]. \tag{18}$$

These balances are satisfied by $[u_{int}] = Ra^{\frac{2}{3}}\,Pr^{-\frac{1}{3}}$

$$\delta_T = Ra^{-\frac{1}{3}}Pr^{\frac{2}{9}}, \quad \delta_H = Ra^{-\frac{1}{3}}Pr^{\frac{5}{9}}, \quad \delta_v = Ra^{-\frac{1}{3}}Pr^{-\frac{1}{9}}$$

As the Prandtl number is increased for a fixed Rayleigh number the velocity boundary layers on the rigid horizontal boundaries thicken and when $Ra^{-\frac{1}{3}}Pr^{\frac{5}{9}} \sim 1$ these boundary layers fill the cell: the boundary layer analysis is then invalid.

Pillow [16] has developed a similar model in which the requirement that the boundary layer vorticity be chosen so as to satisfy the boundary conditions is replaced by a balance between the buoyancy torque and the sheer stress torque. Using this model we find $\delta_H = Pr^{\frac{5}{12}}\,Ra^{-\frac{1}{4}}$ so that a similar breakdown occurs for $Pr^{\frac{5}{12}}\,Ra^{-\frac{1}{4}} \sim 1$.

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## APPENDIX

### *Flow in an infinite Prandtl number viscous fluid with all boundaries free*

A solution to this problem has been outlined by Turcotte and Oxburgh [8]. Here a more complete formulation of certain features of the model as developed in Robinson [6], [7] and the result of a simple computation are given.

The temperature and vorticity equations are equations (1), (3) in the above paper. The temperature boundary conditions are $\psi = \eta = 0$ on all boundaries. Three of the necessary balances (equations (5), (7), (8)) are unchanged by the change of boundary conditions but equation (6) becomes $u \sim [u_{int}]$

$$[u_{int}] = \delta_H^{-2} \qquad \text{(A.1)}$$

since the interior velocity parallel to the horizontal boundaries does not now vanish on those boundaries.

The balances may be satisfied to give

$$[u_{int}] = Ra^{\frac{2}{3}}$$
$$\delta_v = \delta_H = Ra^{-\frac{1}{3}}$$
$$N_u \sim Ra^{\frac{1}{3}}.$$

In the vertical boundary layers it is now found that $\nabla^2\theta \sim (\mathbf{v}.\nabla)\theta$ and the model is therefore valid. This model is in fact almost the same as that for the finite Prandtl number and free boundary conditions with the interior velocity everywhere dominant and the temperature convected smoothly alongside the boundaries and around the corners (see Robinson [5]). However the momentum convection term is now absent and we have a balance between momentum creation and conduction in the vertical boundary layers.

The interior equations are $\theta_{int} = 0$, $\nabla^4\psi_{int} = 0$ with boundary conditions $\psi_{int} = 0$ on all boundaries; $\nabla^2\psi_{int} = 0$ on $z = 0, 1$; $\nabla^2\psi_{int} = -Ra\int \theta dx = f(z)$ on $x = 0$; $\nabla^2\psi_{int} = f(1-z)$ on $x = L$, where the integral is across the appropriate boundary layer. This vorticity boundary condition is obtained by twice integrating the vertical vorticity boundary layer equation and setting $\eta_{BL} + \eta_{int} = 0$ on each vertical boundary.

If $f(z)$ is expanded in a sine series,

$$f(z) = \sum_1^\infty a_n \sin n\pi z$$

(which does not necessarily imply that $f(z) \to 0$ as $z \to 0, 1$), the interior stream function is

$$\psi_{int} = -Ra^{\frac{2}{3}} \sum_{n=1}^{\infty} \frac{a_n}{2\pi n} \sin n\pi z \left\{ \frac{x}{\sinh n\pi L} \times [\cosh n\pi(L-x) + (-1)^n \cosh n\pi x] - L\frac{\sinh n\pi x}{\sinh^2 n\pi L}[1 + (-1)^n \cosh nxL] \right\}. \qquad \text{(A.2)}$$

The temperature equation in the vertical boundary layers may be integrated to give $(\partial\psi_{int}/\partial x)\, f(z) = \text{constant} = A$, which is a statement of the conservation of heat flux in those regions. Inserting the above values of $f(z)$, $\psi_{int}$ into this equation, and equating coefficients of $\sin N\pi z$ $(N = 0, 1, 2 \ldots)$ gives

$$\sum_1^\infty c_n a_n^2 = A \qquad \text{(A.3)}$$

$$\sum_{n=N+1}^{\infty} a_n c_n a_{n-N} + \sum_{n=1} a_n c_n a_{n+N} - \sum_{n=1}^{N-1} \times a_n c_n a_{N-n} = 0 \quad N = 1, 2, 3, \ldots$$

where

$$c_n = \frac{1}{2\pi n} \frac{1}{\sinh^2 n\pi L} \{\sinh n\pi L[\cosh n\pi L + (-1)^n] - n\pi L[1 + (-1)^n \cosh L]\}. \qquad \text{(A.4)}$$

These equations determine the as yet unknown constants up to a factor $\sqrt{A}$ where $A$ is the vertical heat flux in the boundary layer. This constant plays the same part in the analysis that the interior vorticity played in the finite Prandtl number problem, and is determined when the problem is solved completely.

The problem has been solved on a computer taking into consideration only the first term in the expansion of $f(z)$ ($a = 0$ for $n > 1$) and using the approximations as outlined in Robinson [5]. The Nusselt number has a maximum of $Nu = 0{\cdot}19\ Ra^{\frac{1}{3}}$ for a cell width $L = 1{\cdot}3$. Turcott and Oxburgh [8], who made the approximation $f(z)$ = constant, obtained a value of $Nu = 0{\cdot}17\ Ra^{\frac{1}{3}}$ for a cell width $L = 1{\cdot}4$. These estimates may be compared with the value of $Nu = 0{\cdot}28\ Ra^{\frac{1}{3}}$ for a large, but finite, Prandtl number. It is suggested that the difference may be due to the crudity of the above approximations to $f(z)$.

**Résumé**—Un modèle a été élaboré pour décrire le mouvement cellulaire bidimensionnel d'un fluide visqueux avec un nombre de Prandtl de l'ordre de l'unité chauffé par en dessous à des nombres de Rayleigh élevés. Ce modèle est facilement adapté pour décrire un tel écoulement pour un nombre de Prandtl infini pourvu que toutes les frontières soient libres. On a suggéré qu'une bonne approximation du problème physique de l'écoulement entre des frontieres horizontales rigides sera obtenue en posant le nombre de Prandtl égal à l'infini, ignorant ainsi la convection de la quantité de mouvement. Un mouvement cellulaire semblable se produit dans l'écoulement dans un milieu poreux à des nombres de Rayleigh élevés et un modèle semblable a été proposé pour décrire ce mouvement. Les caractéristiques de base du modèle sont esquissées et l'on montre que le modèle ne réussit pas à décrire soit le mouvement d'un fluide à nombre de Prandtl infini entre des frontières parallèles rigides soit celui d'un fluide dans un milieu poreux. Une analyse du modèle pour un nombre de Prandtl fini et élevé et un grand nombre de Rayleigh montre que lorsque le nombre de Prandtl est augmenté, les couches limites dynamiques sur les frontières horizontales rigides s'épaississent, et remplissent éventuellement la cellule, perdant ainsi leur identité de couche limite et provoquant la défaillance du modèle.

**Zusammenfassung**—Es wurde ein Modell entwickelt, um die zweidimensionale Zellbewegung einer viskosen, von unten beheizten Flüssigkeit hoher Rayleighzahl und einer Prandtl-Zahl von der Grössenordnung 1 zu beschreiben. Das Modell ist auch geeignet, Strömungen bei unendlich grosser Prandtl-Zahl und freien Begrenzungen zu beschreiben. Es wurde angenommen, dass eine gute Übereinstimmung zum physikalischen Problem der Strömung zwischen festen, horizontalen Begrenzungen erreicht wird, wenn man die Prandtl-Zahl gleich Unendlich setzt und so die Impuls-Bewegung vernachlässigt.

Eine ähnliche Zellbewegung tritt in einem porösen Körper bei hohen Rayleigh-Zahlen auf. Ein entsprechendes Modell wurde vorgeschlagen, um diese Bewegung zu beschreiben, Die Grundmerkmale des Modells werden erwähnt und es wird gezeigt, dass das Modell entweder bei der Beschreibung der Bewegung einer Flüssigkeit bei unendlich grosser Prandtl-Zahl zwischen parallelen festen Begrenzungen oder bei der Beschreibung eines Minimums in einem porösen Körper versagt. Eine Untersuchung des Modells für grosse endliche Prandtl-Zahlen und grosse Rayleigh-Zahlen zeigt, dass bei Erhöhung der Prandtl-Zahl die Geschwindigkeitsgrenzschichten an den festen, horizontalen Begrenzungen anwachsen und die Zelle womöglich auffüllen. Dadurch verlieren sie ihre Eigenschaft als Grenzschicht, was das Versagen des Modells bewirkt.

**Аннотация**—Разработана модель для описания двухмерного ячеистого движения нагреваемой снизу вязкой жидкости с числом Прандтля порядка единицы при большом числе Релея. Эту модель легко приспособить для описания такого движения при бесконечном числе Прандтля, если все границы свободны. Предполагается, что хорошее приближение к физической задаче течения в жестких горизонтальных границах можно получить, приняв число Прандтля равным бесконечности и пренебрегая таким образом конвекцией. Аналогичное ячеистое движение происходит в пористой среде при больших числах Релея, поэтому можно предположить, что этот случай может быть описан предлагаемой моделью. Дается общее описание этой модели и указано, что она не может быть использована для описания течения жидкости в жестких границах при бесконечном числе Прандтля или жидкости в пористой среде. Анализ этой модели для большого конечного числа Прандтля и большого числа Релея показывает, что при уменьшении числа Прандтля толщина гидродинамического пограничного слоя на жесткой горизонтальной границе увеличивается и иногда наполняет ячейку; таким образом, пограничный слой перестает существовать и модель нарушается.